PHYSICAL REVIEW E VOLUME 47, NUMBER 3 MARCH 1993

BRIEF REPORTS

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than 4 printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.

Transition to a turbulent spectrum in the presence of viscous damping

M. B. Haeri*
Department of Physics, University of California, Los Angeles, Los Angeles, California 90024
(Received 10 September 1992)

The local equation for wave turbulence is used to derive an expression for the minimum energy flux required to both overcome viscous damping and generate a Kolmogorov type of turbulent spectrum. This expression is verified numerically by solving the kinetic equation for the interaction of acoustic waves.

PACS number(s): 47.27.Qb, 52.35.Ra, 63.20.Hp, 47.27.Cn

A system of interacting waves is considered turbulent when it is driven so far off equilibrium that reversible nonlinear processes dominate irreversible linear transport, and when many modes of the system lie within a bandwidth determined by nonlinear processes. If an energy flux (or energy throughput) is injected by a source at a low wave number (k_0) so as to drive the system far off equilibrium, reversible nonlinear processes transport the energy to higher and higher wave numbers until, finally, ordinary viscous damping converts the energy into heat (at wave number k_s). The region between k_0 and k_s (known as the inertial regime) is typically characterized by a steady-state power-law distribution analogous to the Kolmogorov spectrum of vortex turbulence [1]. This power spectrum, $e(\omega)$, which characterizes the stochastic distribution of off-equilibrium propagating waves (wave turbulence), depends upon the dimension and the dispersion law. For gravity waves on the surface of a fluid $e(\omega)\sim 1/\omega^4$, whereas for capillary waves, acoustic waves, and Alfvén waves $e(\omega)\sim 1/\omega^{3/2}$ [2–4]. Wave turbulence also accounts for the spectrum of wind-driven surface waves in the ocean [3] and Alfvén waves driven by the solar wind [5]. A minimum energy flux q_m, supplied by a source, is required to drive the system sufficiently far off equilibrium so that the degree of excitation is large enough to overcome viscous damping. Once this condition is established, an inertial regime characterized by the aforementioned power laws is established.

Figure 1 shows the computer simulation of the approach to wave turbulent states as q increases. These simulations are based on the full kinetic equation for interacting acoustic waves as described below. The hypothesis of locality, as introduced by Kolmogorov [1], was not assumed for the simulations yielding Fig. 1. If locality is assumed, it can be the basis for an analytic derivation of q_m as a function of the dissipation coefficient ν. The hypothesis of locality has been used to determine various aspects of the steady-state power spectrum for fully developed turbulence. By locality we mean that the rate of change of energy at some wave number k (or frequency ω) is due to processes at nearby wave numbers [6] (e.g., $k/2$) and not at the source of energy input located at k_0. Using the hypothesis of locality, we find that $q_m \sim \nu^2$. Our computer simulations, which do not

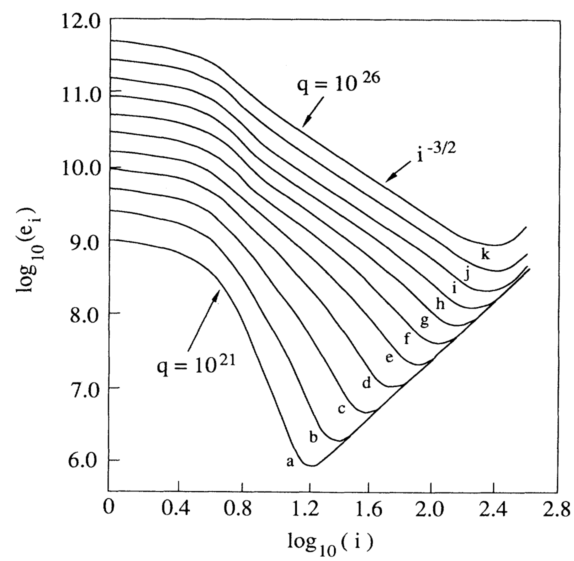

FIG. 1. Turbulent steady-state spectra for $\nu=5\times10^9$. Eleven steady-state spectra were generated for input fluxes ranging from 10^{21} up to 10^{26} ($a=1\times10^{21}$, $b=5\times10^{21}$, $c=1\times10^{22}$, $d=5\times10^{22}$, $e=1\times10^{23}$, $f=5\times10^{23}$, $g=1\times10^{24}$, $h=5\times10^{24}$, $i=1\times10^{25}$, $j=5\times10^{25}$, and $k=1\times10^{26}$). The flux was input over a Gaussian distribution of modes centered at $i=2$. The initial condition used for each case was a Rayleigh-Jeans distribution with $T=2000$. For this dissipation coefficient, $q_m=5\times10^{23}$.

assume locality, yield results in agreement with q_m, as shown in Fig. 2. The Mach number is a key dimensionless parameter for sound defined by the expression $M = v/c$, where v is the fluid velocity and c is the speed of sound. We find that $M_c^2 \sim \nu$, where M_c is the critical Mach number resulting in a turbulent spectrum.

Our interest in acoustic wave turbulence is based upon (i) the possibilities it offers for controlled laboratory experiments, and (ii) the existence of a closed kinetic equation from which one can numerically calculate the time development of the spectral density, thus verifying the results derived from Kolmogorov's local-scaling arguments.

The local equation that describes the rollover of acoustic energy density E_n between frequency regions n and $n+1$ (in region n; $2^n\omega_0 < \omega < 2^{n+1}\omega_0$) is

$$\left[\frac{\partial E_{n+1}}{\partial t}\right]_+ = \frac{\omega_n G^2 E_n^2}{\rho c^2} - \frac{\omega_{n+1} E_{n+1}}{Q} , \tag{1}$$

where ρ is the density and $G[=1+(\rho/c)dc/d\rho]$ is the Gruneisen coefficient that determines the nonlinear coupling of sound modes. The above equation describes the rate at which three-wave processes cause energy in region n to rollover into region $n+1$ via frequency doubling (the subscript $+$ on the time derivative indicates that we are calculating the rate of change due to processes occurring at frequencies up to region $n+1$). According to the principle of locality, the dominant interactions should involve waves with nearly equal wave numbers. So for quadratic nonlinearity, the major consequence of the interactions is to generate a frequency-doubling cascade. In addition to the three-wave process, which is proportional to E_n^2, Eq. (1) includes the usual linear viscous damping as described by the quality factor $Q = \omega/\nu k^2$. For this case a turbulent spectrum will result when the degree of excitation

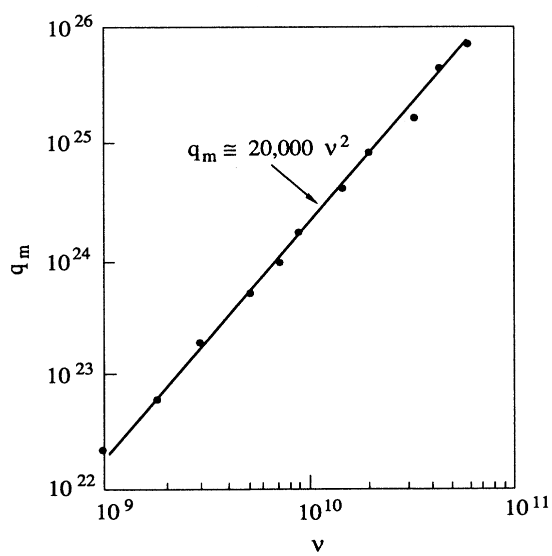

FIG. 2. Relation between q_m and ν. The minimum flux q_m resulting in a turbulent spectrum is plotted as a function of the dissipation coefficient. As shown $q_m \approx \nu^2$.

is high enough to overcome damping or when the condition

$$\frac{E_n}{\rho c^2} \gg \frac{1}{Q_n G^2} \tag{2}$$

or

$$M_n^2 \gg \frac{1}{Q_n G^2} \tag{3}$$

is satisfied. Expressions (2) and (3) are correct to within a numerical constant. In Eq. (3), M_n is the Mach number in the region n, which, in terms of E_n, corresponds to

$$\frac{E_n}{\rho c^2} \cong M_n^2 , \tag{4}$$

where the symbol $\cong$ implies equality except for a numerical factor. In the limit where (2) applies, we can neglect the damping and search for a steady-state solution where the rate of energy rollover q (which is also the rate of energy throughput or flux) is independent of n so that $E_n \cong (\rho c^2 q)^{1/2}/|G|\omega_n^{1/2}$. The steady-state spectral density $e(\omega)$ is then given by

$$e(\omega) = \frac{A(\rho c^2 q)^{1/2}}{\sqrt{\pi}|G|\omega^{3/2}} , \tag{5}$$

where $A(\approx 0.138)$ is the acoustic Kolmogorov constant [7]. Using the definition for Q, Eq. (5), and the fact that

$$E_n = \int_{2^n\omega_0}^{2^{n+1}\omega_0} e(\omega)d\omega , \tag{6}$$

the minimum flux q_m can be calculated by setting $E_n/\rho c^2 = 1/Q_n G^2$ to yield

$$q_m = \beta \frac{\pi\omega_0^3 \rho \nu^2}{c^2 G^2 A^2} , \tag{7a}$$

where β is a numerical constant and ω_0 is the input frequency. The expression for the critical Mach number is then given by

$$M_c \cong \frac{(\beta^{1/2}\omega_0 \nu)^{1/2}}{c|G|} . \tag{7b}$$

When $q > q_m$ there exists an inertial regime where the turbulent spectrum follows a $-\frac{3}{2}$ power law. On the other hand, if $q < q_m$, viscous damping will dominate over the entire frequency range, preventing the formation of a turbulent spectrum that is characterized by a power-law solution.

The off-equilibrium kinetic equation for three-wave interactions is solved to verify Eq. (7a) and to calculate the coefficient β. The kinetic equation that we use is the Peierls [8], Landau-Khalatnikov [9] phonon-Boltzmann equation for an isotropic continuum in the limit where Planck's constant goes to zero. This kinetic equation can also be derived in its own right from classical fluid mechanics [10] or in a more general form from the classical theory of nonlinear coupled oscillators [11]. In dimensionless discrete form, the kinetic equation for the time development of the action I is [7]

$$\frac{\partial I_i}{\partial t}=\sum_{j=1}^{i-1} j^2(i-j)^2[I_jI_{i-j}-I_i(I_{i-j}+I_j)] +2\sum_{j=1}^{N-i} j^2(i+j)^2[I_{i+j}(I_i+I_j)-I_iI_j] +q_0k_0^{-3}\delta_{i,k_0}-\nu i^2\left[I_i-\frac{N}{i}\right], \tag{8}$$

In Eq. (8) we have introduced the dimensionless variables and characteristic action I:

$$\bar{t}=\frac{tG^2\Delta^5\mathcal{J}}{2\pi c^5\rho},\quad \mathcal{J}=\frac{k_BT}{N\Delta},\quad \bar{\nu}=\frac{2\pi\rho c^3\nu}{\mathcal{J}G^2\Delta^3},$$
$$\bar{q}=\frac{q4\pi^3c^8\rho}{G^2\Delta^9\mathcal{J}^2},\quad \bar{I}=\frac{I}{\mathcal{J}},\quad \bar{e}_j=j^3\bar{I}_j, \tag{9}$$

where T is the temperature of the equilibrium state, N is the number of intervals into which the kinetic equation has been discretized, and Δ is the width of each frequency interval [$\omega_j=\Delta_j$ and $I_j=I(j\Delta)$]. In writing Eq. (8) (and in the figures) the bars have been dropped. In this analysis the sink is provided by the viscous damping ν. This method of energy removal is similar to that used in Ref. [12]. Use of a kinetic equation with heat bath-damping requires the introduction of the term N/i, which is the equilibrium value of the action. Fluctuations prevent the action from decaying below this value. The equilibrium solution (when both $q=0$ and $\nu=0$) is a Rayleigh-Jeans distribution corresponding to the equipartition of energy.

The kinetic equation (8) was numerically solved for $N=400$ on an Alliant FX-80 computer. A steady-state spectrum was generated for multiple values of the energy flux ranging from 10^{21} up to 10^{26}, and for several values of the damping coefficient ν ranging from 10^9 up to 6×10^{10}. For each value of ν, steady-state spectra were created for the range of input fluxes. For all cases, the flux was input over a Gaussian distribution of modes centered at $i=2$ (i is the discretized frequency ω). The initial condition for all cases was a Rayleigh-Jeans distribution with $T=2000$. Figure 1 depicts an example of the numerical experiments. Plotted in Fig. 1 are 11 steady-state spectra; each spectrum was generated with a different flux and the same dissipation coefficient $\nu=5\times10^9$. Plot i, j, and k in Fig. 1 are good examples of steady-state turbulent spectra that follow a $-\frac{3}{2}$ power law for over one and a half decades before joining the Rayleigh-Jeans equilibrium distribution. A set of figures similar to Fig. 1 were used to calculate the minimum flux q_m at which only a few modes follow a $-\frac{3}{2}$ power law. In all cases, when $q_{in}=q_m$, the spectra follow a $-\frac{3}{2}$ power law over modes ranging from $i=7-10$. The minimum flux q_m was plotted as a function of the dissipation coefficient (see Fig. 2) to verify the relation between q_m and ν [Eq. (7a)] and to calculate the constant β. As shown in Fig. 2, the numerically calculated minimum flux is $q_m\cong2\times10^4\nu^2$. Our operational definition of the constant β will be when three to four modes are self-similar or follow a $-\frac{3}{2}$ power law. To calculate this coefficient, we use Eq. (7a), the expression for the dimensionless minimum flux,

$$q_m=\beta\frac{i^3\nu^2}{A^2}, \tag{10}$$

and select a frequency, $i=8.5$, which is centrally located between $i=7$ and 10. Using this operational definition, we find that the constant $\beta\cong0.62$, which is essentially of order 1.

We have found that the above result does not depend on the location or width of the source, provided that the energy flux is input at a low frequency where it is far from the sink (or the high-frequency regime, where viscous damping is large). Finally, similar results are obtained when changing the number of modes from 400 to 2000.

For the example shown in Fig. 1, $q_m=5\times10^{23}$ (curve f). The critical Mach number, M_c, can be calculated using Eqs. (9), (7), and (4). Expression (9) must be used to recover physical values for the flux and dissipation coefficient from the dimensionless variables. Taking $\rho=1.0$ g/cm^3 $c=10^4$ cm/sec, and $\Delta=10^6$ Hz so that $T=20$ mK, a critical (dimensionless) flux of 5×10^{23} corresponds to a critical Mach number of order 10^{-7}. If $q<q_m$, or, equivalently, $M<M_c$, the steady-state distribution does not follow a turbulent power law. However, if $q>q_m$, there exists an inertial regime in frequency space that follows the $-\frac{3}{2}$ power law. As q is increased, this inertial region will expand to cover over a decade before rejoining the initial Rayleigh-Jeans distribution. For the case where $q\gg q_m$ (e.g., $q=10^{26}$ in Fig. 1), the turbulent power spectrum no longer joins the initial distribution at high frequency. Instead, the power spectrum joins a Rayleigh-Jeans distribution having a higher temperature.

In this paper we have derived an expression for the minimum flux q_m required to overcome dissipation so that the degree of excitation is high enough to form a turbulent spectrum following a self-similar power-law solution. We have numerically solved the kinetic equation to verify the relation between q_m and ν and calculate the constant β. An experiment designed to study the steady-state spectrum of a turbulent system can also be used to observe this transitional region for acoustic turbulence.

The author would like to thank Seth Putterman for insightful discussions and for paying careful attention to the manuscript. The author is also grateful to the Mathematics Department at UCLA for generous time allocations on their Alliant FX-80 machine. This research was supported by the Hughes Aircraft Company.

*Permanent address: Hughes Aircraft Co., Electro-Optical Systems, P.O. Box 902, El Segundo, CA 90245.

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 301 (1941).

[2] V. E. Zakharov, in *Basic Plasma Physics,* edited by M. N. Rosenbluth and R. Z. Sagdeev (Elsevier, New York, 1984), Vol 2, Ch. 5.1, p. 3; R. Kraichnan, Phys. Fluids **11**, 265 (1968); R. Z. Sagdeev, Rev. Mod. Phys. **51**, 1 (1979).

[3] A. Larraza, S. L. Garret, and S. Putterman, Phys. Rev. A **41**, 3144 (1990).

[4] A. Larraza, Ph.D. thesis, UCLA, 1987; A. Larraza and S. Putterman, in *Irreversible Phenomena and Dynamical Systems Analysis in Geosciences,* edited by C. Nicolis and G. Nicolis (Reidel, Dordrecht, 1987), p. 139.

[5] C. T. Russel, in *Solar Wind,* edited by C. P. Sonett, P. J. Coleman, Jr., and J. M. Wilcox, NASA Report No. SP-308 (1972), pp. 365–374 (unpublished).

[6] U. Frisch, Ann. N.Y. Acad. Sci. **357**, 359 (1980).

[7] M. B. Haeri, A. Garcia, S. J. Putterman, and P. H. Roberts (unpublished).

[8] R. Peierls, *Quantum Theory of Solids* (Clarendon, Oxford, 1955).

[9] L. D. Landau and I. M. Khalatnikov, in *Collected Papers of L. D. Landau,* edited by D. ter Haar (Pergamon, Oxford, 1965), p. 494.

[10] D. J. Benney and P. G. Saffman, Proc. R. Soc. London, Ser. A **289**, 301 (1966).

[11] S. J. Putterman and P. H. Roberts, Phys. Rep. **168**, 209 (1988).

[12] S. L. Musher, Phys. Lett. **70A**, 361 (1979).